Absolute Extreme Values

Recall from Calculus I: If the function $f(x)$ is continuous on an interval $[a, b]$, then the function attains both an *absolute minimum value*, m, and an *absolute maximum value*, M, on that interval. (This is known as the **Extreme Value Theorem**.) In other words, there exist values x_1 and x_2 in the interval such that $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for all $x \in [a, b]$. The values m and M are referred to as the **absolute extreme values** of the function for the interval.

The interval $[a, b]$ is a **bounded** and **closed** set of real numbers. It is bounded because it has finite length. In contrast, an interval such as $[0, \infty)$ is *unbounded* (it has infinite length). $[a, b]$ is closed because it includes its *boundary points* (i.e., its endpoints). In contrast, the interval $(0, 5)$ is not closed (although it is bounded).

A continuous function may not have absolute extreme values on an interval that is either unbounded or not closed. For example, the function $f(x) = x^3$ does not have absolute extreme values on the interval $(-\infty, \infty)$, and the function $f(x) = \tan x$ does not have absolute extreme values on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Suppose $f(x)$ is continuous on $[a, b]$. If x^* is a point in $[a, b]$ at which f attains an absolute extreme value, then x^* must be either a boundary point (i.e., an endpoint) of the interval or a *critical value* (i.e., a value of x for which $f'(x)$ is either zero or undefined). Thus, to find the function's absolute minimum value m and absolute maximum value M for the interval, evaluate the function at a, at b, and at every critical value between a and b. The smallest result must be m and the largest result must be M.

Example: Let $f(x) = x^3 - x^2 - 5x$, and let $[a, b] = [0, 4]$. $f'(x) = 3x^2 - 2x - 5 = (3x - 5)(x + 1)$, so $\frac{5}{3}$ and -1 are critical values. Only the former belongs to $[0, 4]$. We now evaluate the function:

- $f(0) = 0$
- $f(\frac{5}{3}) = -\frac{175}{27} \approx -6.48$
- $f(4) = 28$

Thus, we have $m = -\frac{175}{27}$, occurring at $x = \frac{5}{3}$, and $M = 28$, occurring at $x = 4$.

Now let us adapt the above theory to a function with a two-dimensional domain.

The Extreme Value Theorem: For the function $f(x, y)$, let S be a subset of its domain that is closed and bounded. (Saying it is closed means it includes its boundary points. Saying it is bounded means it can be contained within a circle of finite radius.) If $f(x, y)$ is continuous on S, then the function attains both an *absolute minimum value*, m, and an *absolute maximum value*, M, on S. In other words, there exist points (x_1, y_1) and (x_2, y_2) in S such that $f(x_1, y_1) = m$, $f(x_2, y_2) = M$, and $m \le f(x, y) \le M$ for all $(x, y) \in S$.

If (x^*, y^*) is a point in S at which f attains an absolute extreme value, then (x^*, y^*) must be either a boundary point of S or a *critical point* (i.e., a point for which ∇f is either zero or undefined). Thus, to find the function's absolute minimum value m and absolute maximum value M for S, evaluate the function at at every critical point in S, and also find the function's extreme values on its boundary. The smallest result must be m and the largest result must be M.

Problem 1: Let $f(x,y) = x^2 - 2xy + 2y$, and let $S = [0,3] \times [0,2]$. f is continuous on S because it is a polynomial function. $\nabla f = < 2x - 2y, -2x + 2 >$, which is never undefined and which is $\mathbf{0}$ at only one critical point, $(1,1)$, which does lie in S. $f(1,1) = 1$. Now we must find the function's extreme values on its boundary. S is a closed rectangle with four edges:

1. The vertical line segment with endpoints $(0,0)$ and $(0,2)$, which is a segment of the vertical line $x = 0$. Along this line, $f(x,y) = 2y$. For $y \in [0,2]$, the minimum value is 0 (occurring when $y = 0$) and the maximum value is 4 (occurring when $y = 2$).
2. The horizontal line segment with endpoints $(0,2)$ and $(3,2)$, which is a segment of the horizontal line $y = 2$. Along this line, $f(x,y) = x^2 - 4x + 4$. For $x \in [0,3]$, the minimum value is 0 (occurring when $x = 2$) and the maximum value is 4 (occurring when $x = 0$).
3. The vertical line segment with endpoints $(3,0)$ and $(3,2)$, which is a segment of the vertical line $x = 3$. Along this line, $f(x,y) = 9 - 6y + 2y = -4y + 9$. For $y \in [0,2]$, the minimum value is 1 (occurring when $y = 2$) and the maximum value is 9 (occurring when $y = 0$).
4. The horizontal line segment with endpoints $(0,0)$ and $(3,0)$, which is a segment of the horizontal line $y = 0$. Along this line, $f(x,y) = x^2$. For $x \in [0,3]$, the minimum value is 0 (occurring when $x = 0$) and the maximum value is 9 (occurring when $x = 3$).

Thus, on the boundary of S, f attains a minimum value of 0, occurring at $(0,0)$ and $(2,2)$, and a maximum value of 9, occurring at $(3,0)$. These values are lower and higher than the value 1 ocurring at the critical point. Consequently, on the set S itself, f attains a minimum value of 0, occurring at $(0,0)$ and $(2,2)$, and a maximum value of 9, occurring at $(3,0)$.

In the next problem, we consider an application of these concepts to a relatively simple engineering task. However, the mathematics is surprisingly complicated. Also, the problem does not fit neatly into the framework we have so far considered...

Problem 2: A cardboard box has the shape of a rectangular box without a lid. The total area of cardboard used to construct the box (i.e., the surface area of the box) is 12 square meters. Find the dimensions that maximize the volume of the box.

Solution:

Say the box has length x, width y, and height z. Then its volume is $V = xyz$ and its surface area is $S = xy + 2xz + 2yz$. Setting the latter equal to 12 and solving for z, we get $z = \frac{12 - xy}{2x + 2y}$. Substituting this in place of z in the formula for V, we get $V(x,y) = xy\frac{12 - xy}{2x + 2y} = \frac{12xy - x^2y^2}{2x + 2y}$.

x, y, and z must be positive. It follows that $12 - xy > 0$, $xy < 12$, $y < \frac{12}{x}$. The domain of V is thus $\{(x,y) \mid x > 0 \text{ and } 0 < y < \frac{12}{x}\}$. This region is neither closed nor bounded, so the

Extreme Value Theorem is not applicable. Nonetheless, it can be shown that V does attain an absolute maximum value in this region. The absolute max must occur at a critical point, where $\nabla V = \mathbf{0}$ or is undefined. It cannot occur at a boundary point, because the region's boundary points are excluded from the region (i.e., it is an open region).

To find V_x and V_y, we must use the Quotient Rule.
$V_x = \frac{(12y-2xy^2)(2x+2y)-(12xy-x^2y^2)(2)}{(2x+2y)^2} = \frac{24xy+24y^2-4x^2y^2-4xy^3-24xy+2x^2y^2}{4(x+y)^2} = \frac{24y^2-2x^2y^2-4xy^3}{4(x+y)^2} = \frac{2y^2(12-x^2-2xy)}{4(x+y)^2} = \frac{y^2(12-x^2-2xy)}{2(x+y)^2}$. By similar analysis, $V_y = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$

For either partial derivative, since x and y are positive, the denominator is never zero. Hence, on the domain of V, ∇V is never undefined.

Each partial derivative is equal to zero when its numerator is equal to zero.

- For $V_x = 0$, we have $y^2 = 0$ or $12 - x^2 - 2xy = 0$. We rule out the former since y is positive.
- For $V_y = 0$, we have $x^2 = 0$ or $12 - 2xy - y^2 = 0$. We rule out the former since x is positive.

Matching $12 - x^2 - 2xy = 0$ with $12 - 2xy - y^2 = 0$, we obtain the equation $12 - x^2 - 2xy = 12 - 2xy - y^2$, or $x^2 - y^2 = 0$, or $(x-y)(x+y) = 0$, implying $x = y$ or $x = -y$. We rule out the latter since x and y are both positive. Hence $x = y$.

If $x = y$, the equation $12 - x^2 - 2xy = 0$ becomes $12 - x^2 - 2x^2 = 0$, or $12 - 3x^2 = 0$, or $3(2-x)(2+x) = 0$. Since x is positive, we must have $x = 2$, hence $y = 2$ and $z = \frac{8}{8} = 1$.

Furthermore, if $x = y$, the equation $12 - 2xy - y^2 = 0$ becomes $12 - 2x^2 - x^2 = 0$, or $12 - 3x^2 = 0$, yielding the same solution.

So $(2,2)$ is the only critical point in the domain of V.

We could confirm that V has a relative maximum at $(2,2)$ by means of the Second Derivative Test, but this involves finding V_{xx}, V_{yy}, and V_{xy}, which are complicated in this case. Alternatively, we could examine the surface $z = \frac{12xy - x^2y^2}{2x+2y}$ graphically, using computer software or a sufficiently powerful graphing calculator, to confirm this.

Generally speaking, a relative maximum is not necessarily an absolute maximum, but it can be, and in this case it is. This can be seen by examining the surface $z = \frac{12xy - x^2y^2}{2x+2y}$ graphically, but it can also be inferred on the basis that V is continuous on its domain and has exactly one local extreme value, occurring at a *unique* critical point.

Hence, the maximum volume is $V(2,2) = \frac{12(2)(2) - 2^2 2^2}{2(2)+2(2)} = \frac{48-16}{8} = \frac{32}{8} = 4$. We could also obtain this value simply by using the dimensions previously determined (i.e., length 2, width 2, height 1) and computing length times width times height.